

# ON $\Lambda$ -CONVERGENCE ALMOST EVERYWHERE OF MULTIPLE TRIGONOMETRIC FOURIER SERIES<sup>1</sup>

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**Abstract:** We consider one type of convergence of multiple trigonometric Fourier series intermediate between the convergence over cubes and the  $\lambda$ -convergence for  $\lambda > 1$ . The well-known result on the almost everywhere convergence over cubes of Fourier series of functions from the class  $L(\ln^+ L)^d \ln^+ \ln^+ \ln^+ L([0, 2\pi)^d)$  has been generalized to the case of the  $\Lambda$ -convergence for some sequences  $\Lambda$ .

**Key words:** Trigonometric Fourier series, Rectangular partial sums, Convergence almost everywhere.

Suppose that  $d$  is a natural number,  $\mathbb{T}^d = [-\pi, \pi)^d$  is a  $d$ -dimensional torus, and  $\varphi: [0, +\infty) \rightarrow [0, +\infty)$  is a nondecreasing function. Let  $\varphi(L)(\mathbb{T}^d)$  be the set of all Lebesgue measurable real-valued functions  $f$  on the torus  $\mathbb{T}^d$  such that

$$\int_{\mathbb{T}^d} \varphi(|f(\mathbf{t})|) d\mathbf{t} < \infty.$$

Let  $f \in L(\mathbb{T}^d)$ ,  $\mathbf{k} = (k^1, k^2, \dots, k^d) \in \mathbb{Z}^d$ ,  $\mathbf{x} = (x^1, x^2, \dots, x^d) \in \mathbb{R}^d$ , and  $\mathbf{k}\mathbf{x} = k^1x^1 + k^2x^2 + \dots + k^dx^d$ . Denote by

$$c_{\mathbf{k}} = \frac{1}{(2\pi)^d} \int_{\mathbb{T}^d} f(\mathbf{t}) e^{-i\mathbf{k}\mathbf{t}} d\mathbf{t}$$

the  $\mathbf{k}$ th Fourier coefficient of the function  $f$  and by

$$\sum_{\mathbf{k} \in \mathbb{Z}^d} c_{\mathbf{k}} e^{i\mathbf{k}\mathbf{x}} \tag{1}$$

the multiple trigonometric Fourier series of the function  $f$ .

Let  $\mathbf{n} = (n^1, n^2, \dots, n^d)$  be a vector with nonnegative integer coordinates, and let  $S_{\mathbf{n}}(f, \mathbf{x})$  be the  $\mathbf{n}$ th rectangular partial sum of series (1):

$$S_{\mathbf{n}}(f, \mathbf{x}) = \sum_{\mathbf{k}=(k^1, \dots, k^d): |k^j| \leq n^j, 1 \leq j \leq d} c_{\mathbf{k}} e^{i\mathbf{k}\mathbf{x}}.$$

Denote by  $\text{mes} E$  the Lebesgue measure of a set  $E$  and let  $\ln^+ u = \ln(u + e)$ ,  $u \geq 0$ .

In 1915, in the case  $d = 1$ , N.N. Luzin (see [1]) suggested that the trigonometric Fourier series of any function from  $L^2(\mathbb{T})$  converges almost everywhere. A.N. Kolmogorov [2] constructed an example of a function  $F \in L(\mathbb{T})$  whose trigonometric series diverges almost everywhere and, later on [3], of a function from  $L(\mathbb{T})$  with the Fourier series divergent everywhere on  $\mathbb{T}$ . L. Carleson [4] proved that Luzin's conjecture is true: if  $f \in L^2(\mathbb{T})$ , then the Fourier series of the function  $f$  converges almost

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everywhere. R. Hunt [5] generalized the statement about the almost everywhere convergence of the Fourier series to the class  $L(\ln^+ L)^2(\mathbb{T})$ , particularly, to  $L^p(\mathbb{T})$  with  $p > 1$ . P. Sjölin [6] generalized it to the wider class  $L(\ln^+ L)(\ln^+ \ln^+ L)(\mathbb{T})$ . In [7], the author showed that the condition  $f \in L(\ln^+ L)(\ln^+ \ln^+ \ln^+ L)(\mathbb{T})$  is also sufficient for the almost everywhere convergence of the Fourier series of the function  $f$ . At present, the best negative result in this direction belongs to S.V. Konyagin [8]: if a function  $\varphi(u)$  satisfies the condition  $\varphi(u) = o(u\sqrt{\ln u / \ln \ln u})$  as  $u \rightarrow +\infty$ , then, in the class  $\varphi(L)(\mathbb{T})$ , there exists a function with the Fourier series divergent everywhere on  $\mathbb{T}$ .

Let us now consider the case  $d \geq 2$ , i.e., the case of multiple Fourier series. Let  $\lambda \geq 1$ . A multiple Fourier series of a function  $f$  is called  $\lambda$ -convergent at a point  $\mathbf{x} \in \mathbb{T}^d$  if there exists a limit

$$\lim_{\min\{n^j: 1 \leq j \leq d\} \rightarrow +\infty} S_{\mathbf{n}}(f, \mathbf{x})$$

considered only for vectors  $\mathbf{n} = (n^1, n^2, \dots, n^d)$  such that  $1/\lambda \leq n^i/n^j \leq \lambda$ ,  $1 \leq i, j \leq d$ . The  $\lambda$ -convergence is called the convergence over cubes (the convergence over squares for  $d = 2$ ) in the case  $\lambda = 1$  and the Pringsheim convergence in the case  $\lambda = +\infty$ , i. e., in the case without any restrictions on the relation between coordinates of vectors  $\mathbf{n}$ .

N.R. Tevzadze [9] proved that, if  $f \in L^2(\mathbb{T}^2)$ , then the Fourier series of the function  $f$  converges over cubes almost everywhere. Ch. Fefferman [10] generalized this result to functions from  $L^p(\mathbb{T}^d)$ ,  $p > 1$ ,  $d \geq 2$ . P. Sjölin [11] showed that, if a function  $f$  is from the class  $L(\ln^+ L)^d(\ln^+ \ln^+ L)(\mathbb{T}^d)$ ,  $d \geq 2$ , then its Fourier series converges over cubes almost everywhere. The author [12] (see also [13]) proved the almost everywhere convergence over cubes of Fourier series of functions from the class  $L(\ln^+ L)^d(\ln^+ \ln^+ \ln^+ L)(\mathbb{T}^d)$ . The best current result concerning the divergence over cubes on a set of positive measure of multiple Fourier series of functions from  $\varphi(L)(\mathbb{T}^d)$ ,  $d \geq 2$ , belongs to S.V. Konyagin [14]: for any function  $\varphi(u) = o(u(\ln u)^{d-1} \ln \ln u)$  as  $u \rightarrow +\infty$ , there exists a function  $F \in \varphi(L)(\mathbb{T}^d)$  with the Fourier series divergent over cubes everywhere.

On the other hand, Ch. Fefferman [15] constructed an example of a continuous function of two variables, i. e., a function from  $C(\mathbb{T}^2)$  whose Fourier series diverges in the Pringsheim sense everywhere on  $\mathbb{T}^2$ . M. Bakhbukh and E.M. Nikishin [16] proved that there exists  $F \in C(\mathbb{T}^2)$  such that its modulus of continuity satisfies the condition  $\omega(F, \delta) = O(\ln^{-1}(1/\delta))$  as  $\delta \rightarrow +0$  and its Fourier series diverges in the Pringsheim sense almost everywhere. A.N. Bakhvalov [17] established that, for  $m \in \mathbb{N}$  and any  $\lambda > 1$ , there is a function  $F \in C(\mathbb{T}^{2m})$  such that the Fourier series of  $F$  is  $\lambda$ -divergent everywhere and the modulus of continuity of  $F$  satisfies the condition

$$\omega(F, \delta) = O(\ln^{-m}(1/\delta)), \quad \delta \rightarrow +0. \quad (2)$$

Later on, Bakhvalov [18] proved the existence of a function  $F \in C(\mathbb{T}^{2m})$  satisfying condition (2) and such that its Fourier series is  $\lambda$ -divergent for all  $\lambda > 1$  simultaneously.

Let  $\Lambda = \{\lambda_\nu\}_{\nu=1}^\infty$  be a nonincreasing sequence of positive numbers. Assume that

$$\Omega_\Lambda = \left\{ \mathbf{n} = (n^1, n^2, \dots, n^d) \in \mathbb{N}^d : \frac{1}{1 + \lambda_{n^i}} \leq \frac{n^i}{n^j} \leq 1 + \lambda_{n^j}, \quad 1 \leq i, j \leq d \right\}.$$

We will say that a multiple Fourier series of a function  $f \in L(\mathbb{T}^d)$  is  $\Lambda$ -convergent at a point  $\mathbf{x} \in \mathbb{T}^d$  if there exists a limit

$$\lim_{\mathbf{n} \in \Omega_\Lambda, \min\{n^j: 1 \leq j \leq d\} \rightarrow \infty} S_{\mathbf{n}}(f, \mathbf{x}).$$

Let us note that, if  $\lambda_\nu \equiv \lambda - 1$  for some  $\lambda > 1$ , then the condition of  $\Lambda$ -convergence turns into the condition of  $\lambda$ -convergence defined above. And if  $\lambda_\nu \rightarrow 0$  as  $\nu \rightarrow \infty$ , then the condition of  $\Lambda$ -convergence is weaker than the condition of  $\lambda$ -convergence for any  $\lambda > 1$ .

The author proved [19] that, if a sequence  $\Lambda = \{\lambda_\nu\}_{\nu=1}^\infty$  satisfies the condition  $\ln^2 \lambda_\nu = o(\ln \nu)$  as  $\nu \rightarrow \infty$ , then there exists a function  $F \in C(\mathbb{T}^2)$  such that its Fourier series is  $\Lambda$ -divergent almost everywhere on  $\mathbb{T}^2$ .

In the present paper, we obtain the following statement that strengthens the result of [12].

**Theorem 1.** *Assume that a nonincreasing sequence of positive numbers  $\Lambda = \{\lambda_\nu\}_{\nu=1}^\infty$  satisfies the condition*

$$\lambda_\nu = O\left(\frac{1}{\nu}\right) \quad (3)$$

*and a function  $\varphi: [0, +\infty) \rightarrow [0, +\infty)$  is convex on  $[0, +\infty)$  and such that  $\varphi(0) = 0$ ,  $\varphi(u)u^{-1}$  increases on  $[u_0, +\infty)$ , and  $\varphi(u)u^{-1-\delta}$  decreases on  $[u_0, +\infty)$  for some  $u_0 \geq 0$  and any  $\delta > 0$ . Assume that the trigonometric Fourier series of any function  $g \in \varphi(L)(\mathbb{T})$  converges almost everywhere on  $\mathbb{T}$ . Then, for any  $d \geq 2$ , the Fourier series of any function  $f$  from the class  $\varphi(L)(\ln^+ L)^{d-1}(\mathbb{T}^d)$  is  $\Lambda$ -convergent almost everywhere on  $\mathbb{T}^d$ .*

Theorem 1 and the result of paper [7] imply the following statement.

**Theorem 2.** *Let a nonincreasing sequence of positive numbers  $\Lambda = \{\lambda_\nu\}_{\nu=1}^\infty$  satisfy condition (3),  $d \geq 2$ . Then the Fourier series of any function  $f$  from the class*

$$L(\ln^+ L)^d(\ln^+ \ln^+ \ln^+ L)(\mathbb{T}^d)$$

*is  $\Lambda$ -convergent almost everywhere on  $\mathbb{T}^d$ .*

**P r o o f** of Theorem 1. Let a sequence  $\Lambda = \{\lambda_\nu\}_{\nu=1}^\infty$  and a function  $\varphi$  satisfy the conditions of the theorem. Let  $\varphi_d(u) = \varphi(u)(\ln^+ u)^{d-1}$  for short. Without loss of generality, we can consider only functions  $\varphi_d$  such that the functions  $\varphi_d(\sqrt{u})$  are concave on  $[0, +\infty)$ . Otherwise, we can consider the functions  $\varphi_d(u + a_d) - b_d$  (with appropriate constants  $a_d$  and  $b_d$ ) instead of  $\varphi_d$ . The corresponding class  $\varphi_d(L)(\mathbb{T}^d)$  will be the same in this case.

Denote by  $S_n(f, \mathbf{x})$  the  $n$ th cubic partial sum of the Fourier series of the function  $f$ :

$$S_n(f, \mathbf{x}) = S_{\mathbf{n}}(f, \mathbf{x}), \quad \text{where } \mathbf{n} = (n, \dots, n).$$

Suppose that

$$M(f, \mathbf{x}) = \sup_{n \in \mathbb{N}} |S_n(f, \mathbf{x})|,$$

$$M_\Lambda(f, \mathbf{x}) = \sup_{\mathbf{n} \in \Omega_\Lambda} |S_{\mathbf{n}}(f, \mathbf{x})|.$$

Under the conditions of the theorem (see [12, formula (3.1) and Lemma 3]), there are constants  $K_d > 0$  and  $y_d \geq 0$  such that

$$\text{mes} \left\{ \mathbf{x} \in \mathbb{T}^d : M(f, \mathbf{x}) > y \right\} \leq \frac{K_d}{y} \left( \int_{\mathbb{T}^d} \varphi_d(|f(\mathbf{x})|) d\mathbf{x} + 1 \right), \quad y > y_d, \quad f \in \varphi_d(L)(\mathbb{T}^d). \quad (4)$$

Using (4), we will prove that, for every  $y > y_d$  and  $f \in \varphi_d(L)(\mathbb{T}^d)$ ,

$$\text{mes} \left\{ \mathbf{x} \in \mathbb{T}^d : M_\Lambda(f, \mathbf{x}) > y \right\} \leq \frac{A_d}{y} \left( \int_{\mathbb{T}^d} \varphi_d(|f(\mathbf{x})|) d\mathbf{x} + 1 \right) \quad (5)$$

and, for every  $f \in \varphi_{d+1}(L)(\mathbb{T}^d)$ ,

$$\int_{\mathbb{T}^d} M_{\Lambda}(f, \mathbf{x}) d\mathbf{x} \leq B_d \left( \int_{\mathbb{T}^d} \varphi_{d+1}(|f(\mathbf{x})|) d\mathbf{x} + 1 \right), \quad (6)$$

where  $A_d$  is independent of  $f$  and  $y$ ;  $B_d$  is independent of  $f$ .

The proof is by induction on  $d$ . Consider the base case, i. e.,  $d = 1$ : statement (5) immediately follows from (4) because  $M(f, \mathbf{x}) = M_{\Lambda}(f, \mathbf{x})$  in the one-dimensional case. Similarly, (6) is a consequence of [5, Theorem 2].

Let  $d \geq 2$ . Suppose that statements (5) and (6) hold for  $d - 1$  and let us show that the same is true for  $d$ .

First, let us prove the validity of (5). Let  $\mathbf{n} = (n^1, n^2, \dots, n^d) \in \Omega_{\Lambda}$ . According to (3), there is an absolute constant  $C > 0$  such that  $\lambda_{\nu} \nu \leq C$  for all natural numbers  $\nu$ . Combining this with the definition of  $\Omega_{\Lambda}$ , we obtain that, for all  $i, j \in \{1, 2, \dots, d\}$ ,

$$|n^i - n^j| \leq C. \quad (7)$$

Recall that, if  $\mathbf{n} = (n^1, n^2, \dots, n^d)$ , then the following representation holds for the  $\mathbf{n}$ th rectangular partial sum of the Fourier series of the function  $f$ :

$$S_{\mathbf{n}}(f, \mathbf{x}) = \frac{1}{\pi^d} \int_{\mathbb{T}^d} \prod_{j=1}^d D_{n^j}(t^j) f(x^1 + t^1, \dots, x^d + t^d) dt^1 \dots dt^d, \quad (8)$$

where  $D_n(t) = \sin((n + 1/2)t)/(2 \sin(t/2))$  is the one-dimensional Dirichlet kernel of order  $n$ . Let us add to and subtract from the  $d$ -dimensional Dirichlet kernel  $\prod_{j=1}^d D_{n^j}(t^j)$  of order  $\mathbf{n}$  the sum

$$\sum_{k=2}^d \left( \prod_{j=1}^k D_{n^1}(t^j) \prod_{j=k+1}^d D_{n^j}(t^j) \right)$$

(here and in what follows, we suppose that all products  $\prod$  with an upper index less than a lower one are equal to 1). Rearranging the terms, we obtain

$$\begin{aligned} \prod_{j=1}^d D_{n^j}(t^j) &= \sum_{k=1}^{d-1} \left( \prod_{j=1}^k D_{n^1}(t^j) \prod_{j=k+1}^d D_{n^j}(t^j) - \prod_{j=1}^{k+1} D_{n^1}(t^j) \prod_{j=k+2}^d D_{n^j}(t^j) \right) + \prod_{j=1}^d D_{n^1}(t^j) = \\ &= \sum_{k=2}^d \left( \prod_{j=1}^{k-1} D_{n^1}(t^j) \prod_{j=k+1}^d D_{n^j}(t^j) \left( D_{n^k}(t^k) - D_{n^1}(t^k) \right) \right) + \prod_{j=1}^d D_{n^1}(t^j). \end{aligned}$$

From this and (8), it follows that

$$\begin{aligned}
S_{\mathbf{n}}(f, \mathbf{x}) &= \sum_{k=2}^d \frac{1}{\pi^d} \int_{\mathbb{T}^d} \left( \prod_{j=1}^{k-1} D_{n^1}(t^j) \prod_{j=k+1}^d D_{n^j}(t^j) (D_{n^k}(t^k) - D_{n^1}(t^k)) \right) \times \\
&\times f(x^1 + t^1, \dots, x^d + t^d) dt^1 \dots dt^d + \frac{1}{\pi^d} \int_{\mathbb{T}^d} \prod_{j=1}^d D_{n^1}(t^j) f(x^1 + t^1, \dots, x^d + t^d) dt^1 \dots dt^d = \\
&= \sum_{k=2}^d \frac{1}{\pi^d} \int_{\mathbb{T}} (D_{n^k}(t^k) - D_{n^1}(t^k)) \times \\
&\times \left( \int_{\mathbb{T}^{d-1}} \prod_{j=1}^{k-1} D_{n^1}(t^j) \prod_{j=k+1}^d D_{n^j}(t^j) f(x^1 + t^1, \dots, x^d + t^d) dt^1 \dots dt^{k-1} dt^{k+1} \dots dt^d \right) dt^k + S_{n^1}(f, \mathbf{x}).
\end{aligned} \tag{9}$$

Note that the latter term on the right hand side of (9) is the  $n^1$ th cubic partial sum of the Fourier series of the function  $f$ . By (7), for all  $k \in \{2, 3, \dots, d\}$  and  $t \in \mathbb{T}$ , we have  $|D_{n^k}(t) - D_{n^1}(t)| \leq C$ . Combining this with (9), we obtain

$$\begin{aligned}
|S_{\mathbf{n}}(f, \mathbf{x})| &\leq \sum_{k=2}^d \frac{C}{\pi^d} \int_{\mathbb{T}} \left| \int_{\mathbb{T}^{d-1}} \prod_{j=1}^{k-1} D_{n^1}(t^j) \prod_{j=k+1}^d D_{n^j}(t^j) \times \right. \\
&\times f(x^1 + t^1, \dots, x^{k-1} + t^{k-1}, t^k, x^{k+1} + t^{k+1}, \dots, x^d + t^d) dt^1 \dots dt^{k-1} dt^{k+1} \dots dt^d \Big| dt^k + |S_{n^1}(f, \mathbf{x})|.
\end{aligned}$$

Applying the definitions of  $M_{\Lambda}(f, \mathbf{x})$  and  $M(f, \mathbf{x})$ , from the latter estimate, we obtain

$$\begin{aligned}
M_{\Lambda}(f, \mathbf{x}) &\leq M(f, \mathbf{x}) + \frac{C}{\pi} \sum_{k=2}^d \int_{\mathbb{T}} \sup_{\mathbf{n}=(n^1, n^2, \dots, n^d) \in \Omega_{\Lambda}} \left| \frac{1}{\pi^{d-1}} \int_{\mathbb{T}^{d-1}} \prod_{j=1}^{k-1} D_{n^1}(t^j) \prod_{j=k+1}^d D_{n^j}(t^j) \times \right. \\
&\times f(x^1 + t^1, \dots, x^{k-1} + t^{k-1}, t^k, x^{k+1} + t^{k+1}, \dots, x^d + t^d) dt^1 \dots dt^{k-1} dt^{k+1} \dots dt^d \Big| dt^k = \\
&= M(f, \mathbf{x}) + \frac{C}{\pi} \sum_{k=2}^d M_k(f, \mathbf{x}),
\end{aligned} \tag{10}$$

where  $M_k(f, \mathbf{x})$  denotes the  $k$ th term of the sum on the left hand side of the equality in (10). Let  $k \in \{2, 3, \dots, d\}$ . Consider  $M_k(f, \mathbf{x})$ . Denote by  $g_{k,t^k}$  the function of  $d-1$  variables that can be obtained from the function  $f$  by fixing the  $k$ th variable  $t^k$ :

$$g_{k,t^k}(t^1, \dots, t^{k-1}, t^{k+1}, \dots, t^d) = f(t^1, \dots, t^{k-1}, t^k, t^{k+1}, \dots, t^d), \quad (t^1, \dots, t^{k-1}, t^{k+1}, \dots, t^d) \in \mathbb{T}^{d-1}.$$

Define  $\Omega'_{\Lambda}$  as the set of  $\mathbf{m}_k = (m^1, \dots, m^{k-1}, m^{k+1}, \dots, m^d) \in \mathbb{N}^{d-1}$  such that  $\mathbf{m} = (m^1, \dots, m^d) \in \Omega_{\Lambda}$ . Note that, in view of the invariance of  $\Omega_{\Lambda}$  with respect to a rearrangement of variables, the set  $\Omega'_{\Lambda}$  is independent of  $k$ . Suppose that  $\mathbf{n}'_k = (n^1, \dots, n^1, n^{k+1}, \dots, n^d) \in \mathbb{N}^{d-1}$ . Then

$$\begin{aligned}
&\frac{1}{\pi^{d-1}} \int_{\mathbb{T}^{d-1}} \prod_{j=1}^{k-1} D_{n^1}(t^j) \prod_{j=k+1}^d D_{n^j}(t^j) \times \\
&\times f(x^1 + t^1, \dots, x^{k-1} + t^{k-1}, t^k, x^{k+1} + t^{k+1}, \dots, x^d + t^d) dt^1 \dots dt^{k-1} dt^{k+1} \dots dt^d =
\end{aligned}$$

$$= S_{\mathbf{n}'_k} \left( g_{k,t^k}, (x^1, \dots, x^{k-1}, x^{k+1}, \dots, x^d) \right)$$

and

$$M_k(f, \mathbf{x}) = \int_{\mathbb{T}} \sup_{\mathbf{n}'_k \in \Omega'_\Lambda} \left| S_{\mathbf{n}'_k} \left( g_{k,x^k}, (x^1, \dots, x^{k-1}, x^{k+1}, \dots, x^d) \right) \right| dx^k.$$

Further,

$$\begin{aligned} \text{mes} \left\{ \mathbf{x} \in \mathbb{T}^d : M_k(f, \mathbf{x}) > y \right\} &= 2\pi \text{mes} \left\{ (x^1, \dots, x^{k-1}, x^{k+1}, \dots, x^d) \in \mathbb{T}^{d-1} : M_k(f, \mathbf{x}) > y \right\} \leq \\ &\leq \frac{2\pi}{y} \int_{\mathbb{T}^{d-1}} M_k(f, \mathbf{x}) dx^1 \dots dx^{k-1} dx^{k+1} \dots dx^d = \\ &= \frac{2\pi}{y} \int_{\mathbb{T}^d} \sup_{\mathbf{n}'_k \in \Omega'_\Lambda} \left| S_{\mathbf{n}'_k} \left( g_{k,x^k}, (x^1, \dots, x^{k-1}, x^{k+1}, \dots, x^d) \right) \right| d\mathbf{x} = \\ &= \frac{2\pi}{y} \int_{\mathbb{T}} \left( \int_{\mathbb{T}^{d-1}} \sup_{\mathbf{n}'_k \in \Omega'_\Lambda} \left| S_{\mathbf{n}'_k} \left( g_{k,x^k}, (x^1, \dots, x^{k-1}, x^{k+1}, \dots, x^d) \right) \right| dx^1 \dots dx^{k-1} dx^{k+1} \dots dx^d \right) dx^k. \end{aligned} \quad (11)$$

From this, applying the induction hypothesis (more precisely, statement (6) for the dimension  $d-1$ ) to the inner integral on the right hand part of (11), we obtain

$$\begin{aligned} \text{mes} \left\{ \mathbf{x} \in \mathbb{T}^d : M_k(f, \mathbf{x}) > y \right\} &\leq \frac{2\pi}{y} \int_{\mathbb{T}} \left( B_{d-1} \int_{\mathbb{T}^{d-1}} \varphi_d(|f(\mathbf{x})|) dx^1 \dots dx^{k-1} dx^{k+1} \dots dx^d + 1 \right) dx^k \leq \\ &\leq \frac{(2\pi)^2 B_{d-1}}{y} \left( \int_{\mathbb{T}^d} \varphi_d(|f(\mathbf{x})|) d\mathbf{x} + 1 \right). \end{aligned} \quad (12)$$

According to (10),

$$\left\{ \mathbf{x} \in \mathbb{T}^d : M_\Lambda(f, \mathbf{x}) > y \right\} \subset \left\{ \mathbf{x} \in \mathbb{T}^d : M(f, \mathbf{x}) > \frac{y}{2} \right\} \cup \left( \bigcup_{k=2}^d \left\{ \mathbf{x} \in \mathbb{T}^d : M_k(f, \mathbf{x}) > \frac{\pi y}{2(d-1)C} \right\} \right). \quad (13)$$

Combining (13), (4) and (12), we obtain (5) with the constant  $A_d = 2K_d + 8\pi(d-1)^2 B_{d-1} C$ .

Now, we only need to prove the validity of statement (6). To this end, let us use statement (5) proved above.

From (5), it follows that the majorant  $M_\Lambda(f, \mathbf{x})$  is finite almost everywhere on  $\mathbb{T}^d$  for all  $f \in \varphi_d(L)(\mathbb{T}^d)$ , in particular, for all  $f \in L^2(T^d)$ . Applying Stein's theorem on limits of sequences of operators [20, Theorem 1], we see that the operator  $M_\Lambda(f, \cdot)$  is of weak type  $(2, 2)$ , i.e., there is a constant  $A_d^2 > 0$  such that, for all  $y > 0$  and  $f \in L^2(T^d)$ ,

$$\text{mes} \left\{ \mathbf{x} \in \mathbb{T}^d : M_\Lambda(f, \mathbf{x}) > y \right\} \leq \frac{A_d^2}{y^2} \int_{\mathbb{T}^d} |f(\mathbf{x})|^2 d\mathbf{x}. \quad (14)$$

Similarly, from [20, Theorem 3], we can obtain the following refinement of statement (5): there is a constant  $\bar{A}_d > 0$  such that, for all  $y \geq \bar{y}_d/2 = \bar{A}_d$  and  $f \in \varphi_d(L)(\mathbb{T}^d)$ ,

$$\text{mes} \left\{ \mathbf{x} \in \mathbb{T}^d : M_\Lambda(f, \mathbf{x}) > y \right\} \leq \int_{\mathbb{T}^d} \varphi_d \left( \frac{\bar{A}_d |f(\mathbf{x})|}{y} \right) d\mathbf{x} \leq \frac{\bar{A}_d}{y} \int_{\mathbb{T}^d} \varphi_d(|f(\mathbf{x})|) d\mathbf{x}. \quad (15)$$

Further, let  $f \in \varphi_d(L)(\mathbb{T}^d)$  and  $y > 0$ . Suppose that

$$g(x) = g_y(x) = \begin{cases} f(x), & |f(x)| > y, \\ 0, & |f(x)| \leq y; \end{cases} \quad h(x) = h_y(x) = f(x) - g(x).$$

Define  $\lambda_f(y) = \text{mes} \{ \mathbf{x} \in \mathbb{T}^d : M_\Lambda(f, \mathbf{x}) > y \}$ . Then

$$\lambda_f(y) \leq \text{mes} \{ \mathbf{x} \in \mathbb{T}^d : M_\Lambda(g, \mathbf{x}) > y/2 \} + \text{mes} \{ \mathbf{x} \in \mathbb{T}^d : M_\Lambda(h, \mathbf{x}) > y/2 \} = \lambda_g(y/2) + \lambda_h(y/2).$$

From this, using the equality

$$\int_{\mathbb{T}^d} M_\Lambda(f, \mathbf{x}) \, d\mathbf{x} = - \int_0^\infty y \, d\lambda_f(y) = \int_0^\infty \lambda_f(y) \, dy$$

(see, for example, [21, Chapter 1, § 13, formula (13.6)]), we obtain

$$\int_{\mathbb{T}^d} M_\Lambda(f, \mathbf{x}) \, d\mathbf{x} \leq \bar{y}_d (2\pi)^d + \int_{\bar{y}_d}^\infty \lambda_f(y) \, dy \leq \bar{y}_d (2\pi)^d + \int_{\bar{y}_d}^\infty \lambda_g\left(\frac{y}{2}\right) \, dy + \int_{\bar{y}_d}^\infty \lambda_h\left(\frac{y}{2}\right) \, dy. \quad (16)$$

Taking into account that  $g \in \varphi_d(L)(\mathbb{T}^d)$  and  $h \in L^\infty(\mathbb{T}^d) \subset L^2(\mathbb{T}^d)$  and applying estimate (15) to  $\lambda_g(y/2)$  and estimate (14) to  $\lambda_h(y/2)$ , from (16), we obtain

$$\begin{aligned} \int_{\mathbb{T}^d} M_\Lambda(f, \mathbf{x}) \, d\mathbf{x} &\leq \bar{y}_d (2\pi)^d + 2\bar{A}_d \int_{\bar{y}_d}^\infty \left( \frac{1}{y} \int_{\mathbb{T}^d} \varphi_d(|g(\mathbf{t})|) \, d\mathbf{t} \right) dy + 4A_d^2 \int_{\bar{y}_d}^\infty \left( \frac{1}{y^2} \int_{\mathbb{T}^d} |h(\mathbf{t})|^2 \, d\mathbf{t} \right) dy = \\ &= \bar{y}_d (2\pi)^d + 2\bar{A}_d \int_{\bar{y}_d}^\infty \left( \frac{1}{y} \int_{\{\mathbf{t} \in \mathbb{T}^d : |f(\mathbf{t})| > y\}} \varphi_d(|f(\mathbf{t})|) \, d\mathbf{t} \right) dy + 4A_d^2 \int_{\bar{y}_d}^\infty \left( \frac{1}{y^2} \int_{\{\mathbf{t} \in \mathbb{T}^d : |f(\mathbf{t})| \leq y\}} |f(\mathbf{t})|^2 \, d\mathbf{t} \right) dy. \end{aligned} \quad (17)$$

Applying Fubini's theorem to the integrals on the right hand side of (17), we conclude that

$$\begin{aligned} \int_{\mathbb{T}^d} M_\Lambda(f, \mathbf{x}) \, d\mathbf{x} &\leq 2\bar{A}_d \int_{\{\mathbf{t} \in \mathbb{T}^d : |f(\mathbf{t})| > \bar{y}_d\}} \varphi_d(|f(\mathbf{t})|) \left( \int_{\bar{y}_d}^{|f(\mathbf{t})|} \frac{dy}{y} \right) d\mathbf{t} + \\ &+ 4A_d^2 \int_{\mathbb{T}^d} |f(\mathbf{t})|^2 \left( \int_{|f(\mathbf{t})|}^\infty \frac{dy}{y^2} \right) d\mathbf{t} + \bar{y}_d (2\pi)^d, \end{aligned}$$

hence, statement (6) follows easily.

Finally, the  $\Lambda$ -convergence of the Fourier series of an arbitrary function from the class  $\varphi_d(L)(\mathbb{T}^d)$  can be obtained from (5) by means of standard arguments (see, for example, [12, Lemma 3]). Theorem 1 is proved.  $\square$

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